

FOLIATIONS MODELLING NONRATIONAL SIMPLICIAL TORIC VARIETIES

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ABSTRACT. We establish a correspondence between simplicial fans, not necessarily rational, and certain foliated compact complex manifolds called LVMB-manifolds. In the rational case, Meersseman and Verjovsky have shown that the leaf space is the usual toric variety. We compute the basic Betti numbers of the foliation for shellable fans. When the fan is in particular polytopal, we can apply El Kacimi's basic version of the hard Lefschetz theorem. This allows us to reformulate Stanley's argument for the positivity of the g -vector: we give a proof that unifies rational and nonrational cases, and avoid singularities altogether.

INTRODUCTION

Correspondence between rational convex polytopes and toric varieties.

The correspondence between rational convex polytopes and projective toric varieties is well-known. It pertains to several fields, including combinatorics, convex geometry, symplectic geometry, algebraic geometry.

Within this picture, simple polytopes correspond to toric varieties that are rationally smooth (i.e., their singularities are of finite-quotient type). We will only consider this restricted correspondence. Stanley used this correspondence to prove Mc Mullen's conjectured conditions on the combinatorics of simple convex polytopes, via the hard Lefschetz theorem for toric varieties [S]. Along with other results by Teissier, Khovanskii and Gromov [Tei, Kho, Gr], this uncovered a fruitful link between convex geometry and algebraic geometry.

On the other hand, Stanley's proof contains hints of some "flaws" in the correspondence. Firstly, simple polytopes come in continuous families (by perturbing the facets' directions), whereas toric varieties, dubbed "frigid crystals" in [Da], do not. The reason is that no toric variety corresponds to a nonrational polytope. A second issue is that the toric variety may not be smooth. In Stanley's proof, the first flaw is easily circumvented, since an arbitrary simple convex polytope can be made rational by a small deformation preserving the combinatorial type. The second one is overcome by using intersection cohomology, which is an overkill [WZ]. Both flaws turn into major difficulties when one tries to generalize these ideas to the nonsimple case (*cf.* Karu's work [K]).

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The non existence problem is solved by Prato in [P], by introducing a generalization of toric orbifolds which is non Hausdorff when the polytope is nonrational.

A generalized correspondence. Relying on works by Meersseman and Verjovsky [M, MV], and by Prato [P] (and also on [Bos, LN, LdM-V, CZ]), we take a new approach by realizing the space corresponding to any simple convex polytope as the leaf space of a smooth foliation. This simultaneously fixes both flaws above: even though we are in principle interested in the leaf space, we lift all statements and proofs to the level of the foliation, where everything is smooth and Hausdorff.

A more accurate formulation of the above is made in terms of fans: recall that to any convex polytope P , we can associate its normal fan Δ , which is complete and polytopal. The map $P \mapsto X$ taking a rational simple convex polytope P to a toric variety X factors out as $P \mapsto \Delta \mapsto X$. The first map is non-injective and the second one is the classical one-to-one correspondence between complete simplicial rational polytopal fans and rationally smooth projective toric varieties (these objects being seen up to isomorphism).

We propose a four-way generalization of the correspondence $\Delta \mapsto X$. The roles of varieties and fans are played, respectively, by *leaf spaces of certain foliations* and suitable *triangulated vector configurations*.

Notice that the correspondence $\Delta \mapsto X$ is based on an implicit choice of a vector configuration: the set of primitive generators of the fan's rays. The four generalizations are, in increasing order of importance from our point of view:

- (1) We allow non-polytopal fans.
- (2) We allow orbifold multiplicities. This amounts to taking nonprimitive generators on rays.
- (3) We have more generators than rays: some vectors in the configuration may not correspond to a ray.
- (4) We do not require rationality of the configuration.

Each of these generalizations has already appeared in the literature. We will simultaneously refine, generalize or desingularize several known constructions, thus giving a unified picture of the smooth, orbifold and nonrational cases.

Related works. We give a simplified account of earlier constructions. Each starts from a convex-geometric object, and (using Gale duality) defines an algebraic- or complex-geometric object. From now on, all fans (resp. polytopes) are simplicial (resp. simple).

A rational fan. To each rational simplicial fan in a vector space $L \otimes_{\mathbb{Z}} \mathbb{R}$, with L a lattice, there corresponds a rationally smooth toric variety X .

A Delzant polytope (necessarily rational). On the symplectic side Delzant proves the existence of a unique symplectic toric manifold in correspondence to each Delzant polytope, i.e., whose normal fan satisfies suitable integrality conditions [D].

A rational polytope and multiplicities attached to facets (equivalently, a rational polytopal fan with multiplicities attached to rays, and a certain height function).

— *Symplectic orbifolds.* Lerman and Tolman generalize Delzant’s theorem to the class of symplectic toric orbifolds, by allowing any rational convex simple polytope and rays generators that are not primitive [LT].

— *Generalized Calabi-Eckmann fibrations I.* Meersseman and Verjovsky prove that toric varieties and “toric varieties with orbifold multiplicities” can be viewed as leaf spaces of the foliated manifolds introduced by Meersseman [MV].

A nonrational polytope, a quasilattice, and rays generators. Prato generalizes the Delzant procedure to any simple convex polytope. A key point is to replace the lattice with a *quasilattice*, i.e., a \mathbb{Z} -module in a vector space, generated by a finite spanning set. For a given simple convex polytope, different choices of a quasilattice and rays generators contained therein yield a family of symplectic spaces called *quasifolds*. When the polytope is rational, this family strictly contains the cases above. When the quasilattice is not a lattice the corresponding spaces are non Hausdorff: nonrationality forces quotient singularities of finite type to become of “infinite type” [P]. We shall refer to these spaces as *toric quasifolds*.

A nonrational polytope. Generalizing the works of López de Medrano and Verjovsky [LdM-V], and of Lœb and Nicolau [LN], Meersseman constructs the family of compact complex manifolds and foliations that corresponds to our polytopal case. To each Z , he associates a non necessarily rational polytope P . He establishes a correspondence between the combinatorial type of P and Z up to deformation [M, Th. 13].

A(n implicit) nonrational fan. Generalizing Meersseman’s construction, Bosio constructs the family of so-called LVMB-manifolds that we consider here (cf. [Bos] and the interpretation in [CZ]).

A stacky fan (equivalently, a rational fan with multiplicities attached to rays).

— *Generalized Calabi-Eckmann fibrations II.* Tambour constructs and studies certain LVMB-manifolds in [Tam]. He discusses the relationship with toric varieties.

— *Stacks.* Another approach for handling the orbifold structure, and turning orbifolds into smooth objects, is to use stacks. We refer to Iwanari’s article [I] for this point of view, initiated by Borisov-Chen-Smith in [BCS].

A nonrational fan. Panov and Ustinovsky construct and study certain LVMB-manifolds in [PU] In the rational case, they discuss the relationship with toric varieties.

Summary of our results. We propose to encode all of the convex-geometric data needed for the construction of a space X —that is, a fan, a choice of a point on each ray and a (quasi)lattice containing each of such points— in a unique and well-studied object: a triangulated vector configuration. We will define two integers b and c that are measures of the nonrationality of the configuration. We consider a family of manifolds with a smooth holomorphic foliation whose topology depends on b and c . The leaf space is, in increasing generality, a smooth toric variety, a toric orbifold or a toric quasifold.

In the latter cases, we see our smoothly foliated manifold as a desingularisation of the leaf space X .

Beyond the construction, our main result is that, in both rational and nonrational cases, the cohomological study can be lifted to the foliation by using basic cohomology. In the case of a shellable fan, we compute the basic Betti numbers. In particular, we show that they only depend on the combinatorial type of the fan (Thm. 2.1). When the fan is polytopal we can apply El Kacimi’s basic version of the hard Lefschetz theorem [EK, 3.4.7]. Thus, we give a version of Stanley’s proof of the positivity of the g -vector featuring a tighter link between convex geometry and complex geometry (Sect. 3.2).

We try to delineate the combinatorial, topological, and convex geometric aspects, each of which being of independent interest. In particular, we emphasize the relevance of convex-geometric methods such as triangulations, shellings and Gale duality. We will develop the symplectic/Kähler point of view in a forthcoming paper. A final remark: even though Connes’s non-commutative geometry gives tools to study leaf spaces of foliations, it seems not to apply here, because, as far as we know, neither Betti numbers nor the hard Lefschetz theorem are available in that theory.

1. CONSTRUCTION

1.1. Triangulated configurations. Let E be an \mathbb{R} -vector space of dimension d .

1.1.1. Vector configurations. A *vector configuration* $V = (v_1, \dots, v_n)$ is a finite, ordered list of vectors, allowing repetitions. We will assume that $\text{Span}_{\mathbb{R}}\{v_1, \dots, v_n\} = E$.

Consider the space of linear relations among v_1, \dots, v_n

$$\text{Rel}(V) := \left\{ a \in \mathbb{R}^n \mid \sum_{1 \leq j \leq n} a_j v_j = 0 \right\},$$

which has dimension $n - d$. We say that a real subspace of \mathbb{R}^n is *rational* when it admits a real basis of vectors in \mathbb{Q}^n (equivalently, \mathbb{Z}^n). We define $b(V)$ as the dimension of the largest rational space contained in $\text{Rel}(V)$, and $c(V)$ as the dimension of the smallest rational space containing $\text{Rel}(V)$. Then $0 \leq b(V) \leq n - d \leq c(V) \leq n$.

The configuration is called *rational* when $\text{Rel}(V)$ is rational or, equivalently, $b(V) = n - d$ or $c(V) = n - d$. Otherwise $2 + b(V) \leq c(V)$, and all such values are possible.

1.1.2. Triangulations. Our main reference for triangulations and related concepts is the book [DL-R-S]. Let $\tau \subset \{1, \dots, n\}$. The *cone over* τ is defined as $\text{cone}(\tau) = \{\sum_{j \in \tau} \mathbb{R}_{\geq 0} v_j\}$. By convention, $\text{cone}(\emptyset) = \{0_E\}$. We say that τ is a *simplex* when the vectors indexed by τ are linearly independent (in particular, pairwise distinct). A *simplicial cone* is a cone over a simplex.

A *triangulation* \mathcal{T} of a configuration V is a collection of simplices such that:

- If $\tau \in \mathcal{T}$ and $\tau' \subset \tau$ then $\tau' \in \mathcal{T}$;
- For all $\tau, \tau' \in \mathcal{T}$, $\text{cone}(\tau) \cap \text{cone}(\tau') = \text{cone}(\tau \cap \tau')$;
- $\bigcup_{\tau \in \mathcal{T}} \text{cone}(\tau) \supset \text{cone}(V)$.

This definition allows that some vectors among v_1, \dots, v_n do not belong to any simplex of \mathcal{T} . We denote by $k \geq 0$ the number of such “ghost vectors”. We will always assume that they are at the end of the list v_1, \dots, v_n . The pair (V, \mathcal{T}) is said to be a *triangulated configuration*.

1.1.3. Relations to other convex-geometric data. Suppose first that a triangulated configuration (V, \mathcal{T}) is given.

Where is the fan? The collection of cones on all of the simplices of \mathcal{T} is a *simplicial fan* Δ , of dimension d . That is a collection of simplicial cones such that: each nonempty face of a cone in Δ is a cone in Δ ; the intersection of any two cones in Δ is a face of each $[Z]$. Notice that the fan Δ does not keep track of the ghost vectors and of the position of the other vectors on their respective rays.

Where is the polytope? In general there is no relevant polytope associated to (V, \mathcal{T}) . In the important special case of Δ being polytopal, there are infinitely many polytopes whose normal fan is Δ , all of the same combinatorial type. Some extra data is needed in order to determine a particular polytope.

Where is the (generalized) lattice? The \mathbb{Z} -submodule of E generated by all the vectors v_1, \dots, v_n is a quasilattice in E . By *lattice* in E we shall mean a quasilattice that is closed, equivalently of rank d . The configuration V being rational is equivalent to Q being a lattice.

Conversely, assume given Prato’s data of: a nonnecessarily rational simple polytope P with h facets; normal vectors v_1, \dots, v_h ; a quasilattice Q containing these vectors. Choose v_{h+1}, \dots, v_n such that v_1, \dots, v_n generate Q . The vectors v_1, \dots, v_h generate the rays of the normal fan Δ of P . This fan determines a triangulation \mathcal{T} on $V = (v_1, \dots, v_n)$ with v_{h+1}, \dots, v_n as ghost vectors.

Actually some information is lost — P can't be recovered from Δ —, but this information is not necessary to build the toric quasifold X as a complex quotient [BP]. As with toric varieties, the benefits of the symplectic reduction construction are an a priori symplectic/Kähler structure and compactness, whereas the advantages of the complex quotient are: an a priori complex structure; a generalization to the non polytopal case. We will give more details later on how to encode and use that extra piece of information, that can exist only in the polytopal case.

Finally, starting from a stacky fan, we encode it in a similar way: we add ghost vectors to generate the ambient lattice, as in [MV].

1.2. Construction of the LVMB-manifold Z .

1.2.1. *Balanced and odd triangulations.* Let (V, \mathcal{T}) be a triangulated vector configuration satisfying:

- (i) $n - d = 2m + 1$ with m a positive integer,
- (ii) $\sum v_i = 0$.

In particular, Δ is a complete fan. On the other hand, these conditions are mild restrictions in the sense that we can always fulfill them while keeping both the quasilattice and the (already complete) fan unchanged (this fact is used in [MV]). Namely, if (V, \mathcal{T}) does not satisfy these two conditions, we apply the following algorithm:

Step 1. If $\sum v_i \neq 0$, append $-\sum v_i$ as a new ghost vector of the configuration (and increase n by 1);

Step 2. If $n - d$ is even, append 0 as a new ghost vector of the configuration (and increase n by 1);

Step 3. If $n - d = 1$, append 0 and 0 as new ghost vectors of the configuration (and increase n by 2).

1.2.2. *Virtual chamber and $U(\mathcal{T})$.* Denote the set of maximal simplices of \mathcal{T} by $\{\mathcal{E}_\alpha\}_\alpha$. Define the *virtual chamber* $\mathcal{E} := \{\mathcal{E}_\alpha^c = \{1, \dots, n\} \setminus \mathcal{E}_\alpha\}_\alpha$. By definition, virtual chambers correspond bijectively to triangulations of V (cf. [AS]). For each α , define $U_\alpha := \{[z_1 : \dots : z_n] \in \mathbb{CP}^{n-1} \mid \forall j \in \mathcal{E}_\alpha^c, z_j \neq 0\}$. Define $U(\mathcal{T}) := \bigcup_\alpha U_\alpha$.

1.2.3. *The dual configuration.* Define a matrix $M \in \mathbb{R}^{n \times (2m+1)}$ by

$$M = \begin{bmatrix} 1 & a_1^1 & \dots & a_1^{2m} \\ & & \ddots & \\ 1 & a_n^1 & \dots & a_n^{2m} \end{bmatrix},$$

where the columns form a basis of $\text{Rel}(V)$. Now define a vector configuration $\hat{\Lambda}^\mathbb{R} = (\hat{\Lambda}_1^\mathbb{R}, \dots, \hat{\Lambda}_n^\mathbb{R})$ in \mathbb{R}^{2m+1} called a *Gale dual* of (v_1, \dots, v_n) , and a

configuration $\Lambda^{\mathbb{R}} = (\Lambda_1^{\mathbb{R}}, \dots, \Lambda_n^{\mathbb{R}})$ in \mathbb{R}^{2m} by

$$M = \begin{bmatrix} -\hat{\Lambda}_1^{\mathbb{R}} & - \\ \vdots & \\ -\hat{\Lambda}_n^{\mathbb{R}} & - \end{bmatrix} = \begin{bmatrix} 1 & -\Lambda_1^{\mathbb{R}} & - \\ & \vdots & \\ 1 & -\Lambda_n^{\mathbb{R}} & - \end{bmatrix}.$$

Notice that M is only defined up to right multiplication by a matrix of form $T = \begin{bmatrix} 1 & B \\ 0 & A \end{bmatrix}$ where $B = (b_1, \dots, b_{2m}) \in \mathbb{R}^{2m}$ and $A \in GL(2m, \mathbb{R})$. Therefore, a Gale dual is not unique, and $\Lambda^{\mathbb{R}}$ is only defined up to the invertible real affine transformation of the ambient \mathbb{R}^{2m} given by $X \mapsto XA + B$. Thus, $\Lambda^{\mathbb{R}}$ is to be seen as a configuration of *points*, i.e., affine objects. We refer to Sect. (4) for examples.

1.2.4. *The \mathbb{C}^m -action and Z .* Consider the holomorphic \mathbb{C}^m -action on $U(\mathcal{T})$ defined by

$$(1) \quad \begin{aligned} \mathbb{C}^m \times U(\mathcal{T}) &\longrightarrow U(\mathcal{T}) \\ (\underline{u}; [z_1 : \dots : z_n]) &\longmapsto [e^{2\pi i \Lambda_1(\underline{u})} z_1 : \dots : e^{2\pi i \Lambda_n(\underline{u})} z_n], \end{aligned}$$

where

$$\Lambda_j := \begin{bmatrix} a_j^1 + ia_j^{m+1} \\ \vdots \\ a_j^m + ia_j^{2m} \end{bmatrix} \in \mathbb{C}^m$$

and a_j^1, \dots, a_j^{2m} denote the entries of $\Lambda_j^{\mathbb{R}}$.

Bosio has given in [Bos] sufficient conditions for this action to be proper and cocompact. We show below that (1) is free and Bosio's conditions hold, thus the quotient of $U(\mathcal{T})$ by this action is a compact complex manifold that we denote Z . If we act on V by a linear automorphism of E , then $\text{Rel}(V)$, $\Lambda^{\mathbb{R}}$, and (therefore) Z are unchanged.

1.2.5. *Proof that Bosio's conditions hold.* By properties of Gale duality (see [DL-R-S] Def. 5.4.3 and the comment below), for each α , $\{\hat{\Lambda}_j^{\mathbb{R}} \mid j \in \mathcal{E}_\alpha^c\}$ is a simplex, i.e., a linear basis of \mathbb{R}^{2m+1} . Let P_α denote the convex hull of $\{\Lambda_j^{\mathbb{R}} \mid j \in \mathcal{E}_\alpha^c\} \subset \mathbb{R}^{2m}$. Then $\mathring{P}_\alpha \neq \emptyset$, and it follows that action (1) has trivial isotropy at any element of U_α , so this action is free.

The result below belongs to a circle of ideas that appear in the works of Białyński-Birula and Świącicka. Similar results include also [BH] Lemma 3.5 and [Tam] Prop. 2.3 and Cor. 2.4.

Proposition 1.1. *Bosio's conditions hold here, i.e.,*

- (i) $\mathring{P}_\alpha \cap \mathring{P}_\beta \neq \emptyset$ for every α, β ;
- (ii) for every $\mathcal{E}_\alpha^c \in \mathcal{E}$ and every $i \in \mathcal{E}_\alpha$,
there exists $k \in \mathcal{E}_\alpha^c$ such that $(\mathcal{E}_\alpha^c \setminus \{k\}) \cup \{i\} \in \mathcal{E}$.

Proof. (i) Pick in \mathcal{T} any two distinct maximal simplices \mathcal{E}_α and \mathcal{E}_β , and choose a linear form φ that separates the respective cones, in the sense that φ is positive on $\text{cone}(\mathcal{E}_\alpha)$ and negative on $\text{cone}(\mathcal{E}_\beta)$, except on $\text{cone}(\mathcal{E}_\alpha) \cap \text{cone}(\mathcal{E}_\beta)$, where it is zero. A linear evaluation such as

$$\left(\varphi(v_1), \dots, \varphi(v_n)\right)$$

corresponds (cf. [DL-R-S] p. 244) to a linear relation on the Gale dual with coefficients given by $\varphi(v_1), \dots, \varphi(v_n)$. Here the relation has the form

$$\sum_{j \in \mathcal{E}_\alpha \setminus \mathcal{E}_\beta} a_j \hat{\Lambda}_j^{\mathbb{R}} - \sum_{j \in \mathcal{E}_\beta \setminus \mathcal{E}_\alpha} b_j \hat{\Lambda}_j^{\mathbb{R}} + \sum_{j \notin \mathcal{E}_\alpha \cup \mathcal{E}_\beta} c_j \hat{\Lambda}_j^{\mathbb{R}} = 0,$$

where all a_j 's and b_j 's are positive. For all $j \notin \mathcal{E}_\alpha \cup \mathcal{E}_\beta$, we write c_j as the difference of two positive numbers $a_j - b_j$. Then

$$\begin{aligned} \sum_{j \in \mathcal{E}_\alpha \setminus \mathcal{E}_\beta} a_j \hat{\Lambda}_j^{\mathbb{R}} + \sum_{j \notin \mathcal{E}_\alpha \cup \mathcal{E}_\beta} a_j \hat{\Lambda}_j^{\mathbb{R}} &= \sum_{j \in \mathcal{E}_\beta \setminus \mathcal{E}_\alpha} b_j \hat{\Lambda}_j^{\mathbb{R}} + \sum_{j \notin \mathcal{E}_\alpha \cup \mathcal{E}_\beta} b_j \hat{\Lambda}_j^{\mathbb{R}}, \text{ i.e.,} \\ \sum_{j \in \mathcal{E}_\beta^c} a_j \hat{\Lambda}_j^{\mathbb{R}} &= \sum_{j \in \mathcal{E}_\alpha^c} b_j \hat{\Lambda}_j^{\mathbb{R}}. \end{aligned}$$

Thus $\sum_{j \in \mathcal{E}_\beta^c} a_j = \sum_{j \in \mathcal{E}_\alpha^c} b_j =: s$, and

$$\frac{1}{s} \sum_{j \in \mathcal{E}_\beta^c} a_j \hat{\Lambda}_j^{\mathbb{R}} = \frac{1}{s} \sum_{j \in \mathcal{E}_\alpha^c} b_j \hat{\Lambda}_j^{\mathbb{R}}.$$

The left hand side and right hand side belong to \dot{P}_β and \dot{P}_α respectively. Therefore the intersection is nonempty.

(ii) Pick $\mathcal{E}_\alpha^c \in \mathcal{E}$ and $i \in \mathcal{E}_\alpha$. The facet of $\text{cone}(\mathcal{E}_\alpha)$ determined by omitting v_i is shared by one and only one maximal cone, say $\text{cone}(\mathcal{E}_\beta)$. Then $\mathcal{E}_\beta = (\mathcal{E}_\alpha \setminus \{i\}) \cup \{k\}$ for some k , and $k \notin \mathcal{E}_\alpha$ by convexity of $\text{cone}(\mathcal{E}_\alpha)$. Then $(\mathcal{E}_\alpha^c \setminus \{k\}) \cup \{i\} = \mathcal{E}_\beta^c \in \mathcal{E}$. \square

1.3. The foliation \mathcal{F} on Z . Consider on $U(\mathcal{T})$ the following holomorphic action by \mathbb{C}^{2m} :

$$(2) \quad t.[z_1 : \dots : z_n] = [e^{2\pi i \Lambda_1^{\mathbb{R}}(t)} z_1 : \dots : e^{2\pi i \Lambda_n^{\mathbb{R}}(t)} z_n].$$

Fix a $[z] \in U(\mathcal{T})$. Direct computations show that the isotropy at $[z]$ is a closed \mathbb{Z} -module $L_z \subset \mathbb{R}^{2m} \subset \mathbb{C}^{2m}$ of rank at most $2m$.

Action (2) commutes with (1), so it descends to Z . The restriction of the action (2) to

$$\mathbb{C}_Z^m := \{t \in \mathbb{C}^{2m} \mid t = \begin{pmatrix} \underline{u} \\ i\underline{u} \end{pmatrix}, \underline{u} \in \mathbb{C}^m\}$$

gives the action (1). Define

$$\mathbb{C}_{\mathcal{F}}^m := \{t \in \mathbb{C}^{2m} \mid t = \begin{pmatrix} v \\ 0 \end{pmatrix}, v \in \mathbb{C}^m\}.$$

The projection $\pi : \mathbb{C}^{2m} = \mathbb{C}_Z^m \oplus \mathbb{C}_{\mathcal{F}}^m \rightarrow \mathbb{C}_{\mathcal{F}}^m$ is given by $(x, y) \mapsto (x + iy, 0)$. The isotropy of $[z] \in Z$ for the action of $\mathbb{C}_{\mathcal{F}}^m$ on Z is $\pi(L_z)$. Therefore this action has discrete isotropy, so it induces on Z a smooth holomorphic foliation \mathcal{F} of dimension m . In the polytopal case, this foliation appears in [LN] and [M] (cases $m = 1$ and $m \geq 1$ respectively).

1.3.1. Leaves. The leaf \mathcal{F}_z through a point $[z] \in Z$ is the image, via an injective immersion, of $\mathbb{C}_{\mathcal{F}}^m / \pi(L_z)$ and is in general not closed. For each simplex $\tau \in \mathcal{T}$ let $\tau^c := \{1, \dots, n\} \setminus \tau$. Consider the subconfiguration $\hat{\Lambda}^{\mathbb{R}}(\tau) = (\hat{\Lambda}_j^{\mathbb{R}})_{j \in \tau^c}$. We define $b(\tau) := \#\tau^c - c(\hat{\Lambda}^{\mathbb{R}}(\tau))$ and $c(\tau) := \#\tau^c - b(\hat{\Lambda}^{\mathbb{R}}(\tau))$. The numbers $b(\tau)$ and $c(\tau)$ only depend on V and τ . Moreover:

$$b(\tau) \leq 2m + 1 \leq c(\tau) \leq \#\tau^c;$$

$$b(\tau) \leq b(\tau') \leq 2m + 1 \leq c(\tau') \leq c(\tau) \text{ whenever } \tau' \supset \tau;$$

$b(\emptyset) = n - c(\hat{\Lambda}^{\mathbb{R}}) = b(V)$ and $c(\emptyset) = n - b(\hat{\Lambda}^{\mathbb{R}}) = c(V)$. The last formulas follow from $\text{Rel}(\hat{\Lambda}^{\mathbb{R}}) = \text{Ker } M^t = (\text{Im } M)^{\perp} = \text{Rel}(V)^{\perp}$.

The holomorphic action of $(\mathbb{C}^*)^{n-1} \simeq (\mathbb{C}^*)^n / \mathbb{C}^*(1, \dots, 1)$ on $\mathbb{C}P^{n-1}$ induces a decomposition of Z (cf. [GMP, Defn. p. 36]): define $Z(\tau) \subset Z$ as the image in Z of $\{[z] \in U(\mathcal{T}) \mid z_j \neq 0 \text{ iff } j \in \tau^c\}$. Then Z is the disjoint union of the $(\mathbb{C}^*)^{n-1}$ -orbits $\coprod_{\tau \in \mathcal{T}} Z(\tau)$, with a unique open orbit $Z(\emptyset)$. For each $[z] \in Z(\tau)$, $\text{rank}(\pi(L_z)) = \text{rank}(L_z)$, since π restricted to \mathbb{R}^{2m} is an isomorphism. If $[z] \in Z(\tau)$ then $\text{rank}(\pi(L_z)) = b(\tau) - 1$, hence the topological type of \mathcal{F}_z is

$$(S^1)^{b(\tau)-1} \times \mathbb{R}^{2m-b(\tau)+1}.$$

In particular, it does not depend on the choice of the Gale dual. On the other hand, different choices of a Gale dual make L_z vary through all possible closed \mathbb{Z} -modules of rank $b(\tau) - 1$ in \mathbb{R}^{2m} : when we multiply M by T given by A, B (cf. Sect. 1.2.3), the \mathbb{Z} -module becomes $A^{-1}L_z$. Therefore the holomorphic structure of \mathcal{F}_z does depend on the choice of the Gale dual. All structures of a complex abelian group of dimension m are possible, i.e., all products of the form $\mathbb{C}^{\alpha} \times (\mathbb{C}^*)^{\beta} \times G$ where G is a Cousin group. If either the configuration is rational or $[z] \in Z(\tau)$, with τ maximal simplex, then the rank of the lattice L_z is $2m$ and therefore the leaf is a compact complex torus (cf. [M, MV]). The number $b(V) = b(\emptyset)$ gives the topological type of the generic leaf. If the configuration is rational, that is $b(V) = 2m + 1$, all leaves are closed. On the other hand there are nonrational configurations V such that $b(V) = 1$, so in this cases the generic leaf is \mathbb{C}^m .

1.3.2. Closure of leaves and space of leaf closures. If $[z] \in Z(\tau)$, then $\overline{\mathcal{F}_z} \simeq (S^1)^{c(\tau)-1}$. In particular the leaf \mathcal{F}_z is closed if and only if the subconfiguration $(\hat{\Lambda}_j^R)_{j \in \tau^c}$ is rational, equivalently $b(\tau) = 2m + 1$ or $c(\tau) = 2m + 1$.

Therefore, if the configuration is rational, that is $c(V) = c(\emptyset) = 2m + 1$, the space of leaf closures is the leaf space itself. On the other hand, when $c(V) = n$, the closure of a generic leaf is an orbit of the maximal compact subgroup $(S^1)^{n-1}$ of $(\mathbb{C}^*)^{n-1}$, acting on Z by biholomorphisms. If the configuration is totally nonrational, that is $c(\tau) = \#\tau^c$ for all τ , then the space of leaf closures is the usual manifold with corners associated with toric varieties.

Consider now the linear map $\mathbb{R}^{2m+1} \rightarrow \mathbb{R}^n$ defined by the matrix M . By theorem [M, Thm. 4], the degree of transcendence of the field of meromorphic functions on Z is greater than or equal to the number $a(V)$ defined as the dimension of the largest rational subspace of $\ker M^t$. On the other hand $\text{Rel}(V) = \text{Im} M = (\ker M^t)^\perp$ implies $a(V) = n - c(V)$.

1.3.3. The leaf space. Let Δ, v_1, \dots, v_h and Q be the fan, rays generators (i.e., non-ghost vectors, so $h = n - k$) and quasilattice associated to (V, \mathcal{T}) (see Sect. 1.1.3). It is known that to these data there corresponds a geometric quotient $X = U(\Delta)/G$, where $U(\Delta)$ is an open subset of \mathbb{C}^h that depends on the combinatorics of Δ , and G is a complex subgroup of $(\mathbb{C}^*)^h$ that depends on Q and on the vectors v_1, \dots, v_h . If the configuration is rational (resp. nonrational), then X is a complex manifold or a complex orbifold (resp. a non Hausdorff complex quasifold) of dimension d , acted on holomorphically by the torus (resp. quasitorus) \mathbb{C}^d/Q (cf. [A, C, P, BP]; the construction in [BP, Thm 2.2] can be adapted to the nonpolytopal case). Let (Z, \mathcal{F}) be any foliated complex manifold corresponding to (V, \mathcal{T}) . The leaf space is given by Z modulo the \mathbb{C}^{2m} -action (2). We define a projection map $p: Z \rightarrow X$. Let $[z_1, \dots, z_n] \in Z = U(\mathcal{T})/\mathbb{C}^m$, then, from Sect. 1.2.5, there exists $\underline{t} \in \mathbb{C}^{2m}$ such that

$$\underline{t} \cdot [z_1, \dots, z_n] = [e^{2\pi i(\Lambda_1^{\mathbb{R}} - \Lambda_n^{\mathbb{R}})(\underline{t})} z_1 z_n^{-1}, \dots, e^{2\pi i(\Lambda_h^{\mathbb{R}} - \Lambda_n^{\mathbb{R}})(\underline{t})} z_h z_n^{-1}, 1, \dots, 1].$$

Then the projection

$$\begin{aligned} p : U(\mathcal{T})/\mathbb{C}^m &\longrightarrow U(\Delta)/G \\ [z_1, \dots, z_n] &\longmapsto [e^{2\pi i(\Lambda_1^{\mathbb{R}} - \Lambda_n^{\mathbb{R}})(\underline{t})} z_1 z_n^{-1}, \dots, e^{2\pi i(\Lambda_h^{\mathbb{R}} - \Lambda_n^{\mathbb{R}})(\underline{t})} z_h z_n^{-1}] \end{aligned}$$

is well defined and identifies the leaf space Z/\mathcal{F} with X as complex quotients. In particular the complex structure induced by (Z, \mathcal{F}) on the leaf space depends on the initial data alone. A local description of the projection p can be deduced from (3) p. 13.

The holomorphic action of the (quasi)torus \mathbb{C}^d/Q on the leaf space X is induced, via the projection p , by the holomorphic action of $(\mathbb{C}^*)^{n-1}$ on Z .

Remark 1.2. The action of the group G does induce a holomorphic foliation on $U(\Delta)$. In the polytopal case, El Kacimi's theorem can be applied in this setting (the noncompactness of $U(\Delta)$ is not an essential obstacle). However, since G is in general, for rational and nonrational configurations, not connected, the leaf space is *not* X . This problem is overcome in our construction by "increasing the dimension".

2. TOPOLOGICAL RESULTS: BASIC BETTI NUMBERS IN THE SHELLABLE CASE

In this section we show how the combinatorics of a balanced and odd triangulated configuration (V, \mathcal{T}) relate to the basic Betti numbers of any foliated manifold (Z, \mathcal{F}) built from (V, \mathcal{T}) . The formulas are the same as the usual Betti numbers of simplicial toric varieties.

For the combinatorial part we refer the reader to [Z, Sect. 8.3]; for basic cohomology, see [Ton]. We recall definitions and results in the form we need for our purposes.

2.1. Shellings and h -vector. Fix a triangulated vector configuration (V, \mathcal{T}) . In particular, \mathcal{T} is an abstract simplicial complex of pure dimension $d - 1$ (topologically, a sphere). The *dimension* of a (possibly empty) simplex $\tau \in \mathcal{T}$ is $\#\tau - 1$. Recall that the *f -vector* $(f_{-1}, f_0, f_1, \dots, f_{d-1})$ records the number of simplices in each dimension. The fan Δ gives a “linear realization” of this simplicial complex, with simplices of dimension l corresponding bijectively to cones of dimension $l + 1$.

A *shelling* of \mathcal{T} (or of Δ) is a linear ordering of the maximal simplices $\mathcal{E}_1, \dots, \mathcal{E}_{f_{d-1}}$ such that for all $\alpha \geq 2$, $\text{cone}(\mathcal{E}_\alpha)$ intersects $\text{cone}(\mathcal{E}_1) \cup \dots \cup \text{cone}(\mathcal{E}_{\alpha-1})$ along a nonempty union of facets of $\text{cone}(\mathcal{E}_\alpha)$. The number of such facets, called the *index* of \mathcal{E}_α w.r.t. the shelling, is denoted i_α . We take $i_1 = 0$.

Polytopal fans are shellable, i.e., they admit a shelling ([Z, Sect. 8.2]). The *h -vector* (h_0, h_1, \dots, h_d) of \mathcal{T} (or Δ) records the number of maximal simplices of each index in a given shelling. It is well-known that the h -vector is completely determined by the f -vector—in particular, it is independent of the choice of a shelling—and conversely it determines the f -vector.

2.2. Basic cohomology. Let M be a smooth manifold with a smooth foliation \mathcal{G} . A differential form $\omega \in \Omega^\bullet(M)$ is called *basic* when for all vector fields X tangent to \mathcal{G} , $\iota_X \omega = 0$ and $\iota_X d\omega = 0$. When the foliation is given by the orbits of a Lie group G , this means that the form is G -invariant and its kernel contains the tangent space to \mathcal{G} . The cohomology of the complex of basic forms is in some sense the de Rham cohomology of the leaf space. The dimensions of these groups are called the *basic Betti numbers*. An example that gives some intuition for the proofs below is the torus $M = S^1 \times S^1$ with \mathcal{G} given by lines of slope s . When s is rational, the leaf space is a circle and $b_{\mathcal{G}}^1(M) = 1$. When s is irrational, the leaf space is not Hausdorff but, again, $b_{\mathcal{G}}^1(M) = 1$. Cohomologically, the leaf space is still a circle.

2.3. Computation of the basic Betti numbers.

Theorem 2.1. *Let (V, \mathcal{T}) be a shellable, balanced and odd triangulated vector configuration, with $\dim(V) = d$. Let (Z, \mathcal{F}) be any foliated manifold built from (V, \mathcal{T}) . Then the basic Betti numbers are*

$$b_{\mathcal{F}}^{2j+1}(Z) = 0$$

and

$$b_{\mathcal{F}}^{2j}(Z) = h_j$$

for $j = 0, \dots, d$.

Proof. We use a “Morse-theoretic” method due to Khovanskii for simple polytopes. Working dually with simplicial fans, we see that his method extends (from polytopal fans) to shellable fans. Let $\mathcal{E}_1, \dots, \mathcal{E}_{f_{d-1}}$ be a shelling of \mathcal{T} . Consider the open subsets U_α defined in 1.2.2 and their image, Z_α , in Z . We consider the \mathcal{F} -saturated open covering of Z defined as follows:

$$\begin{aligned} W_1 &= Z_1, \\ W_\alpha &= W_{\alpha-1} \cup Z_\alpha, \quad \alpha = 2, \dots, f_{d-1}. \end{aligned}$$

Therefore

$$Z_1 = W_1 \subset W_2 \subset \dots \subset W_{f_{d-1}} = Z.$$

We compute inductively the basic cohomology of the foliated manifolds W_α by means of a Mayer-Vietoris sequence. For this we need a *basic partition of unity*: pick any partition of unity relative to the decomposition $W_\alpha = W_{\alpha-1} \cup Z_\alpha$. The natural $(S^1)^d$ -action on \mathbb{C}^d descends to Z . Averaging out the functions over this action will in particular makes them constant on the leaves.

From Lemma 2.2, case $r = 0$, we know that $b_{\mathcal{F}}^l(W_1 = Z_1) = \delta_{0,l}$. Now fix $\alpha \geq 2$ and make the induction hypothesis:

$$(\mathcal{H}_{\alpha-1}) \quad \text{if } l \text{ is odd then } b_{\mathcal{F}}^l(W_{\alpha-1}) = 0.$$

We claim that

$$b_{\mathcal{F}}^l(W_\alpha) = \begin{cases} b_{\mathcal{F}}^l(W_{\alpha-1}) & \text{if } l \neq 2i_\alpha, \\ b_{\mathcal{F}}^l(W_{\alpha-1}) + 1 & \text{if } l = 2i_\alpha. \end{cases}$$

In particular, (\mathcal{H}_α) holds, and the proposition follows by induction.

Now we prove the claim. Using the notation of Lemma 2.2 below, remark first that $W_{\alpha-1} \cap Z_\alpha$ is of the form $Z_{\alpha,\tau}$, where τ is the restriction of \mathcal{E}_α , defined in [Z, 8.3] as $\tau = \{i \in \mathcal{E}_\alpha \mid (\mathcal{E}_\alpha \setminus i) \subset \mathcal{E}_\beta \text{ with } \beta < \alpha\}$. Notice that $\#\tau = i_\alpha$.

Then Lemma 2.2 tells us that $W_{\alpha-1} \cap Z_\alpha$ has no basic cohomology in positive even dimension. Thus by Mayer-Vietoris, for any odd integer p ,

$$\begin{aligned} 0 \rightarrow H_{\mathcal{F}}^p(W_\alpha) &\rightarrow \underbrace{H_{\mathcal{F}}^p(W_{\alpha-1})}_{=0 \text{ by } (\mathcal{H}_{\alpha-1})} \oplus \underbrace{H_{\mathcal{F}}^p(Z_\alpha)}_{=0 \text{ by Lemma 2.2}} \rightarrow \underbrace{H_{\mathcal{F}}^p(W_{\alpha-1} \cap Z_\alpha)}_{(*)} \rightarrow H_{\mathcal{F}}^{p+1}(W_\alpha) \rightarrow \\ &\underbrace{H_{\mathcal{F}}^{p+1}(W_{\alpha-1})}_{=0 \text{ by Lemma 2.2}} \oplus \underbrace{H_{\mathcal{F}}^{p+1}(Z_\alpha)}_{=0 \text{ by Lemma 2.2}} \rightarrow 0 \end{aligned}$$

We see that the second term, $H_{\mathcal{F}}^p(W_\alpha)$, must be zero. Again by Lemma 2.2, $(*)$ is zero unless $p = 2i_\alpha - 1$, in which case it is of dimension one. \square

Lemma 2.2. *Let (V, \mathcal{T}) be a balanced and odd triangulated vector configuration in a vector space of dimension d . Let τ be a subset of a maximal simplex \mathcal{E}_α of \mathcal{T} . We define an \mathcal{F} -saturated open subset of Z , denoted $Z_{\alpha, \tau}$, as the image in Z of*

$$U_{\alpha, \tau} = U_\alpha \setminus \{ [z_1 : \cdots : z_n] \mid \forall j \in \tau, z_j = 0 \}.$$

In particular, $Z_{\alpha, \tau} = Z_\alpha$ when τ is empty. Then, denoting $r = \#\tau$,

$$\forall l \geq 0, H_{\mathcal{F}}^l(Z_{\alpha, \tau}) \approx \begin{cases} \mathbb{R} & \text{if } l = 0 \text{ or } l = 2r - 1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the leaf space $Z_{\alpha, \tau}/\mathcal{F}$ is cohomologically a point when $r = 0$, and a $2r - 1$ -sphere when $r > 0$.

Proof. By definition of \mathcal{F} we have $H_{\mathcal{F}}^l(Z_{\alpha, \tau}) \approx H^l(\Omega_{\mathbb{C}^{2m}}^\bullet(U_{\alpha, \tau}))$, where $\Omega_{\mathbb{C}^{2m}}^\bullet(U_{\alpha, \tau})$ denotes the complex of forms on $U_{\alpha, \tau}$ that are basic with respect to the foliation induced by the \mathbb{C}^{2m} -action (2).

Suppose for now that $r > 0$, and assume for simplicity that $\mathcal{E}_\alpha = \{1, \dots, d\}$ and $\tau = \{1, \dots, r\}$. We know that $\{\Lambda_j^\mathbb{R} - \Lambda_n^\mathbb{R}\}_{j=d+1 \dots n-1}$ is an \mathbb{R} -basis of \mathbb{R}^{2m} , so it is a \mathbb{C} -basis of \mathbb{C}^{2m} . This implies surjectivity of the map

$$g : (\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m} \rightarrow U_{\alpha, \tau} \\ ((z_1, \dots, z_d); \underline{t}) \mapsto \underline{t} \cdot [z_1 : \cdots : z_d : \underbrace{1 : \cdots : 1}_{2m+1}].$$

On the other hand, $g((z_1, \dots, z_d); \underline{t}) = g((w_1, \dots, w_d); \underline{s})$ is equivalent to

$$\begin{cases} (w_1, \dots, w_d) = (e^{2\pi i(\Lambda_1^\mathbb{R} - \Lambda_n^\mathbb{R})(\underline{t} - \underline{s})} z_1, \dots, e^{2\pi i(\Lambda_d^\mathbb{R} - \Lambda_n^\mathbb{R})(\underline{t} - \underline{s})} z_d) \\ (\Lambda_j^\mathbb{R} - \Lambda_n^\mathbb{R})(\underline{t} - \underline{s}) \in \mathbb{Z}, j = d+1 \dots n-1. \end{cases}$$

The second condition implies that $\underline{t} - \underline{s} \in L_{\underline{u}}$ with $[[\underline{u}]]$ any point in $Z(\mathcal{E}_\alpha)$. Therefore $L_{\underline{u}}$ is a lattice in \mathbb{R}^{2m} that we denote Γ . This shows that the fibers of g are the orbits of the Γ -action on $(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m}$ defined by

$$(3) \quad \underline{\gamma} \cdot ((z_1, \dots, z_d); \underline{t}) = ((e^{2\pi i(\Lambda_1^\mathbb{R} - \Lambda_n^\mathbb{R})(\underline{\gamma})} z_1, \dots, e^{2\pi i(\Lambda_d^\mathbb{R} - \Lambda_n^\mathbb{R})(\underline{\gamma})} z_d); \underline{t} - \underline{\gamma}).$$

Notice that the action of Γ on the first factor does not depend on the choice of a Gale dual: changing this choice, $\Lambda_j^\mathbb{R} - \Lambda_n^\mathbb{R}$ becomes $(\Lambda_j^\mathbb{R} - \Lambda_n^\mathbb{R})A$ and Γ becomes $A^{-1}\Gamma$ (cf. 1.2.3 and 1.3.1), thus $(\Lambda_j^\mathbb{R} - \Lambda_n^\mathbb{R})A(A^{-1}\gamma) = (\Lambda_j^\mathbb{R} - \Lambda_n^\mathbb{R})(\gamma)$. Now, $\omega \mapsto g^*(\omega)$ maps isomorphically the complex $\Omega_{\mathbb{C}^{2m}}^\bullet(U_{\alpha, \tau})$ onto the complex \mathcal{C}^Γ of forms on $(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m}$ that are: (a) Γ -invariant; (b) basic with respect to the foliation with leaves $\{\underline{z}\} \times \mathbb{C}^{2m}$. But a form satisfies condition (b) if and only if it is the pull-back of a form by the projection $(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}) \times \mathbb{C}^{2m} \rightarrow \mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}$. Therefore \mathcal{C}^Γ is (isomorphic to) the complex of Γ -invariant forms $\Omega^\bullet(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r})^\Gamma$. By (3), we see that

the action of Γ factors through the standard $(S^1)^d$ -action on \mathbb{C}^d . Therefore we can apply [Ba, Lemma 2.2] to conclude that for every $l \geq 0$,

$$\begin{aligned} H^l\left(\Omega^\bullet(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r})^\Gamma\right) &\approx H^l\left(\Omega^\bullet(\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r})\right) \\ &\approx H^l(\mathbb{C}^r \setminus \{0\}) \approx H^l(S^{2r-1}). \end{aligned}$$

In the case $r = 0$ the proof is similar: replace every $\mathbb{C}^r \setminus \{0\} \times \mathbb{C}^{d-r}$ with \mathbb{C}^d , and omit the last line. \square

After this paper was completed, we came across the article [GT] that contains results in the spirit of our Th. 2.1.

3. STANLEY'S PROOF REVISITED

Let Δ be a polytopal simplicial fan. Denote its h -vector by (h_0, \dots, h_d) , and define its g -vector (g_1, \dots, g_δ) , with $\delta = \lfloor \frac{d}{2} \rfloor$, by $g_j = h_j - h_{j-1}$, $j = 1 \dots \delta$.

Choose a triangulated vector configuration (V, \mathcal{T}) whose associated fan is Δ , and a corresponding (Z, \mathcal{F}) .

The h -vector is known to satisfy the Dehn-Sommerville equations, $h_{d-j} = h_j$ for all j . The g -vector is known to be positive by a result of Stanley [S] (for a review see also [Ful]).

In the spirit of [S], we show below that these combinatorial properties have a direct interpretation in terms of the basic cohomology of (Z, \mathcal{F}) . First of all let us remark that a corollary of Th. 2.1 is that \mathcal{F} is homologically orientable, i.e.,

$$(4) \quad H_{\mathcal{F}}^{2d}(Z) = 1.$$

3.1. Dehn-Sommerville equations. By our computation of the basic Betti numbers in Th. 2.1, these relations are equivalent to Poincaré duality for the basic cohomology of (Z, \mathcal{F}) , which is known to hold for a homologically orientable foliation by [EK, 3.2.8].

As is well-known, the D-S equations follow more simply by computing the h_i 's using a shelling and the reverse shelling [Z, 8.21]. This is in fact a combinatorial abstraction of the Morse-theoretic proof of Poincaré duality.

3.2. Positivity of the g -vector.

Proposition 3.1. *The basic cohomology of (Z, \mathcal{F}) satisfies the hard Lefschetz theorem. In particular, there exists an injective map $H_{\mathcal{F}}^{i-2}(Z) \rightarrow H_{\mathcal{F}}^i(Z)$ for all $i \leq d$.*

Proof. Loeb-Nicolau (for $m = 1$, in [LN]) and Meersseman (for $m \geq 1$, in [M]) have shown that \mathcal{F} is transversely Kähler. Using (4), we apply El Kacimi's hard Lefschetz [EK, 3.4.7] and conclude. \square

Now, we can prove the positivity of the g -vector by applying Stanley's argument to the smoothly foliated manifold corresponding to *any* configuration, including nonrational configurations.

Corollary 3.2. *The g -vector of Δ is positive.*

Proof. Fix j such that $1 \leq j \leq \delta$. By Th. 2.1, $g_j = b_{\mathcal{F}}^{2j}(Z) - b_{\mathcal{F}}^{2j-2}(Z)$, which is positive by Prop 3.1. \square

4. A MODEL EXAMPLE: THE PROJECTIVE PLANE \mathbb{CP}^1 AND VARIANTS

4.1. Let $E = \mathbb{R}$. Consider the configuration $V := (p, -q, q - p, 0)$, where p and q are positive reals. Notice that V is equivalent, by linear isomorphism, to $(\frac{p}{q}, -1, 1 - \frac{p}{q}, 0)$. Therefore, there is only one real parameter here, namely the fraction $\frac{p}{q}$. We will distinguish the following cases

- (a) $\frac{p}{q} \in \mathbb{Q}$ with p, q coprime integers;
- (b) $\frac{p}{q} \in \mathbb{R} \setminus \mathbb{Q}$.

We triangulate V by $\mathcal{T} = \{\mathcal{E}_1 = \{1\}, \mathcal{E}_2 = \{2\}, \emptyset\}$. Then the fan and quasilattice associated with (V, \mathcal{T}) are respectively: the one dimensional fan whose maximal cones are $\text{cone}(\mathcal{E}_1) = \text{Span}_{\mathbb{R}_{\geq 0}} v_1 = \mathbb{R}_{\geq 0}$ and $\text{cone}(\mathcal{E}_2) = \text{Span}_{\mathbb{R}_{\geq 0}} v_2 = \mathbb{R}_{\leq 0}$, that is the usual fan of \mathbb{CP}^1 ; the quasilattice $Q = \text{Span}_{\mathbb{Z}}\{p, q\}$. In case (a) Q is \mathbb{Z} . In case (b), Q is dense and has rank two. The virtual chamber is $\mathcal{E} = \{\mathcal{E}_1^c = \{234\}, \mathcal{E}_2^c = \{134\}\}$ and

$$U(\mathcal{T}) = U_1 \cup U_2 \\ = \{[z] \in \mathbb{CP}^3 \mid z_2 \neq 0, z_3 \neq 0, z_4 \neq 0\} \cup \{[z] \in \mathbb{CP}^3 \mid z_1 \neq 0, z_3 \neq 0, z_4 \neq 0\}.$$

Choose $M = \begin{bmatrix} 1 & q & 0 \\ 1 & p & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ so $\Lambda^{\mathbb{R}} = \begin{bmatrix} q & p & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\Lambda = [q \ p \ 0 \ i]$. Action (1) reads

$$\mathbb{C} \times U(\mathcal{T}) \longrightarrow U(\mathcal{T}) \\ \left(t; [z]\right) \longmapsto [e^{2\pi i q t} z_1 : e^{2\pi i p t} z_2 : z_3 : e^{-2\pi t} z_4],$$

giving a compact complex quotient Z . Action (2) reads

$$\mathbb{C}^2 \times U(\mathcal{T}) \longrightarrow U(\mathcal{T}) \\ \left((t, u); [z]\right) \longmapsto [e^{2\pi i q t} z_1 : e^{2\pi i p t} z_2 : z_3 : e^{2\pi i u} z_4].$$

As noted in 1.3.1, the leaves corresponding to the maximal simplices are closed. The leaf \mathcal{F}_1 corresponding to the simplex $\{1\}$ is given by $\mathbb{C}_{\mathcal{F}}/\pi(L_1) \approx \mathbb{C}/\text{Span}_{\mathbb{Z}}\{\frac{1}{p}, i\}$, while the leaf \mathcal{F}_2 corresponding to the simplex $\{2\}$ is given by $\mathbb{C}_{\mathcal{F}}/\pi(L_2) \approx \mathbb{C}/\text{Span}_{\mathbb{Z}}\{\frac{1}{p}, i\}$.

Now let $[z]$ be a generic point, i.e., $[z] \in Z(\emptyset)$. Then $(t, u) \in L_{[z]} \Leftrightarrow qt, pt, u \in \mathbb{Z}$. In case (a), $L_{[z]} = \mathbb{Z}^2$, so $\mathcal{F}_{[z]} \approx \mathbb{C}/\text{Span}_{\mathbb{Z}}\{1, i\}$. Hence leaves divide into three types, all of them are 2-dimensional tori, endowed with nonequivalent complex structures.

In case (b), $\text{Rank}(L_z) = 0$, the generic leaf is \mathbb{C} and its closure in Z is an $(S^1)^3$

From the proof of Lemma 2.2, we know that $\mathbb{C} \hookrightarrow U_1, z_1 \mapsto [z_1 : 1 : 1 : 1]$ gives a local slice for action (2). This slice is stabilized by $\frac{1}{p}\mathbb{Z} \times \mathbb{Z} \subset \mathbb{C}^2$. Hence, $X_1 := Z_1/\mathcal{F} = U_1/\mathbb{C}^2$ can be identified with \mathbb{C} modulo $z_1 \mapsto e^{2\pi i \frac{q}{p}t} z_1$.

In case (a), the map $\mathbb{C} \rightarrow X_1$ is a local uniformization with cyclic group of order p . Using the slice $\mathbb{C} \hookrightarrow U_2, z_2 \mapsto [1 : z_2 : 1 : 1]$, we obtain another local uniformization $\mathbb{C} \rightarrow X_2$ with cyclic group of order q . If $p = q = 1$ then the leaf space $X = X_1 \cup X_2$ is $\mathbb{C}P^1$. For p, q any coprime integers, the leaf space is a weighted projective space, that is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the action of \mathbb{C}^* with weights q and p .

In case (b), the space X_1 (resp. X_2) can be identified respectively with \mathbb{C} modulo $z_1 \mapsto e^{2\pi i \frac{q}{p}t} z_1$ (resp. $z_2 \mapsto e^{2\pi i \frac{p}{q}t} z_2$). The local groups at the poles are now infinite, of rank one. The leaf space is the toric quasifold described in detail in [P, Ex. 1.13, 3.5] and [BP, Ex. 2.6].

To describe \mathcal{F} , and see how it desingularizes X , compose the above $\mathbb{C} \hookrightarrow U_1$ with the quotient $U_1 \rightarrow Z_1$. We obtain a slice $\mathbb{C} \hookrightarrow Z_1$ for the action of $\mathbb{C}_{\mathcal{F}}$. Each orbit of the local groups above corresponds to the intersection between a leaf of \mathcal{F} and this slice. The leaf passing through $z_1 = 0$ is the leaf \mathcal{F}_1 above; it intersects the slice only once.

In case (a), the leaf through any $z_1 \neq 0$ hits the slice p times, so it wraps p times around \mathcal{F}_1 .

In case (b), the leaf through any $z_1 \neq 0$ hits the slice infinitely many times, so it wraps infinitely many times around \mathcal{F}_1 .

4.2. Now we encode in a vector configuration the case of p and q non necessarily coprime, $Q = \mathbb{Z}$. Choose $V = (p, -q, 1, q - p - 1)$ and take

$$M = \begin{bmatrix} 1 & q & 0 \\ 1 & p & 1 \\ 1 & 0 & q \\ 1 & 0 & 0 \end{bmatrix} \text{ so } \Lambda^{\mathbb{R}} = \begin{bmatrix} q & p & 0 & 0 \\ 0 & 1 & q & 0 \end{bmatrix} \text{ and } \Lambda = [q \quad p+i \quad qi \quad 0].$$

Action (1) becomes $t.[z] = [e^{2\pi i q t} z_1 : e^{2\pi i (p+i)t} z_2 : e^{-2\pi i q t} z_3 : z_4]$ and action (2) becomes $(t, u).[z] = [e^{2\pi i q t} z_1 : e^{2\pi i (pt+u)} z_2 : e^{2\pi i q u} z_3 : z_4]$.

We take the same triangulation \mathcal{T} as above, so we can use the same slices, which are stabilized respectively by $\{(t, u) = (\frac{qk-l}{pq}, \frac{l}{q}) \mid k, l \in \mathbb{Z}\} \subset \mathbb{C}^2$ and $\{(t, u) = (\frac{k}{q}, \frac{l}{q}) \mid k, l \in \mathbb{Z}\} \subset \mathbb{C}^2$. These groups act on the slices by $(k, l).z_1 = e^{2\pi i q t} z_1 = e^{2\pi i \frac{qk-l}{p}} z_1$ and $(k, l).z_2 = e^{2\pi i (pt+u)} z_2 = e^{2\pi i \frac{pk+l}{q}}$. The leaf space is therefore an orbifold with singularities at the poles of order $p, q \in \mathbb{Z}_{\geq 1}$. This is similar to [MV, Ex. 5.3], although our construction does not involve the choice of a Kähler class (we give an interpretation of this extra piece of information in 4.4.2). In conclusion, in the rational case, one

can prescribe at the poles orbifold singularities of arbitrary order p, q , with $p, q \in \mathbb{Z}_{\geq 1}$.

4.3. What happens in the nonrational case? By a suitable choice of vector configuration one can prescribe an *arbitrary* finitely generated subgroup A of the circle as local group at both poles of the corresponding toric quasifold. Assume without loss of generality that A is generated by $e^{2\pi i r_1}, \dots, e^{2\pi i r_{2m-1}}$ with $r_j \in \mathbb{R}$. Let $V = (1, -1, r_1, \dots, r_{2m-1}, -r_1 - \dots - r_{2m-1})$. Therefore the quasilattice $Q = \text{Span}_{\mathbb{Z}}\{1, r_1, \dots, r_{2m-1}\}$. Take

$$M = \begin{bmatrix} 1 & 1 & -r_1 & \dots & \dots & -r_{2m-1} \\ 1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & \dots & 0 \end{bmatrix},$$

so $\Lambda^{\mathbb{R}} = \left(\begin{bmatrix} 1 \\ -r_1 \\ \vdots \\ -r_{2m-1} \end{bmatrix}, e_1, \dots, e_{2m}, 0 \right)$, where e_1, \dots, e_{2m} is the canonical basis of \mathbb{R}^{2m} .

We keep \mathcal{T} as above. The slice $\mathbb{C} \hookrightarrow U_1, z_1 \mapsto [z_1 : 1 : \dots : 1]$ is stabilized by $\mathbb{Z}^{2m-1} \subset \mathbb{C}^{2m}$, which acts by $h \cdot z_1 = e^{2\pi i(r_1 h_2 + \dots + r_{2m-1} h_{2m})} z_1$. At the other pole the local group is also A , which acts on the corresponding slice in the same way.

4.4. Brief account of the polytopal case.

4.4.1. *Preliminaries.* An important special case is when \mathcal{T} is a *regular triangulation*. Regularity has several characterisations:

1. The triangulation is *regular*
- \Leftrightarrow 2. The fan Δ is *polytopal*
- \Leftrightarrow 3. There exists a *height function* on V that induces \mathcal{T}
- \Leftrightarrow 4. The virtual chamber defines a *chamber*, i.e., $\bigcap_{\alpha} \mathring{P}_{\alpha} \neq \emptyset$ (cf. 1.2.5)
- \Leftrightarrow 5. There exists $\nu \in \mathbb{R}^{2m}$ s.t. $\forall \tau \subset \{1 \dots n\}, \tau \in \mathcal{T}$ if and only if ν is in the interior of the convex hull of $\left\{ \Lambda_j^{\mathbb{R}} \mid j \in \tau^c \right\}$

The last condition implies that, by definition, the corresponding manifold Z is an LVM-manifold [M]. This in turn implies that the foliation \mathcal{F} is transversely Kähler [LN] (for $m = 1$) and [M] (for $m \geq 2$) and that there is a diffeomorphic embedding of the manifold Z into $\mathbb{C}P^{n-1}$.

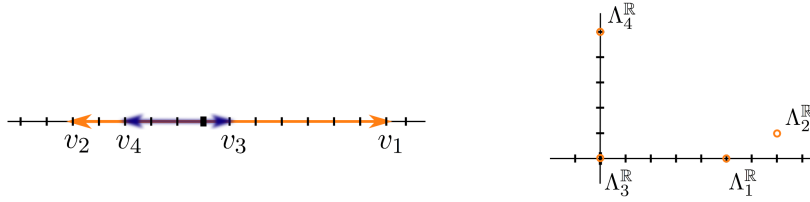
Correspondence between regular triangulations and chambers. Regular triangulations of V are in one-to-one correspondence with *chambers* of $\Lambda^{\mathbb{R}}$, i.e., bounded connected components of $\mathbb{R}^{2m} - L$, where L is the union of all affine $2m - 1$ -planes determined by $\Lambda_1^{\mathbb{R}}, \dots, \Lambda_n^{\mathbb{R}}$. Explicitly: from \mathcal{T} we obtain the chamber $\bigcap_{\alpha} \overset{\circ}{P}_{\alpha}$; from a chamber C , we define \mathcal{T} by $\tau \in \mathcal{T} \Leftrightarrow$ the convex hull of $\{\Lambda_j^{\mathbb{R}} \mid j \in \tau^c\}$ contains C .

Correspondence between height functions and points in a chamber. A triangulation is *regular* when there exists a so-called *height function* $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ such that: there is a (necessarily unique) convex function $\psi_{\omega} : \mathbb{R}^d \rightarrow \mathbb{R}$, restricting to pairwise distinct linear forms on the maximal cones of Δ , such that $\psi_{\omega}(v_i) = \omega_i$ for each non ghost vector v_i and $\psi_{\omega}(v_i) < \omega_i$ for each ghost vector v_i .

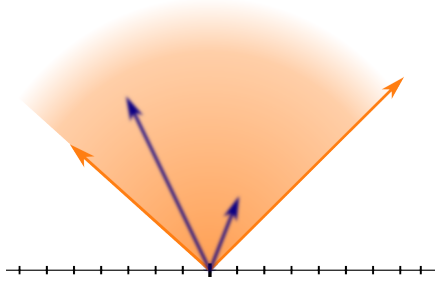
By a result of Carl Lee [DL-R-S, Lemma 5.4.4], ω induces a triangulation \mathcal{T} if and only if $\nu := \frac{1}{\sum \omega_i} \sum \omega_i \Lambda_i^{\mathbb{R}}$ belongs to the chamber associated to \mathcal{T} as above. Therefore, conversely, starting from a point in a chamber written as a convex linear combination $\sum \omega_i \Lambda_i^{\mathbb{R}}$, we obtain a height function ω inducing a regular triangulation.

Conclusion: the map $\omega \mapsto \nu$ gives a quantitative refinement of the qualitative correspondence between regular triangulations and chambers described above.

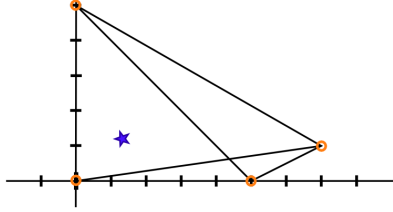
4.4.2. *Example.* We choose the same data as in 4.2 (but here p, q can be any positive reals): $V = (p, -q, 1, q - p - 1)$, $\Lambda^{\mathbb{R}} = \begin{bmatrix} q & p & 0 & 0 \\ 0 & 1 & q & 0 \end{bmatrix}$.



To induce the triangulation $\mathcal{T} = \{\mathcal{E}_1 = \{1\}, \mathcal{E}_2 = \{2\}, \emptyset\}$, we can choose, for example, $\omega_1 = p$, $\omega_2 = q$ (so $\psi_{\omega} = |\cdot|$), $\omega_3 > 1$ and $\omega_4 > |q - p - 1|$:



which in turns gives the point $\nu = \frac{1}{\omega_1 + \omega_2 + \omega_3 + \omega_4}(\omega_1 \Lambda_1^{\mathbb{R}} + \omega_2 \Lambda_2^{\mathbb{R}} + \omega_3 \Lambda_3^{\mathbb{R}} + \omega_4 \Lambda_4^{\mathbb{R}})$ contained in one of the four chambers of the configuration $\Lambda^{\mathbb{R}}$:



Using [M], this point can be used to give a \mathcal{C}^∞ embedding $Z \hookrightarrow \mathbb{CP}^{n-1}$ as

$$\mathcal{Z} = \left\{ [z] \in \mathbb{CP}^{n-1} \mid \sum_{j=1 \dots n} (\Lambda_j^{\mathbb{R}} - \nu) |z_j|^2 = 0 \right\}.$$

This solves a problem mentioned in [MV, Rem. 4.11]. Pulling-back the Fubini-Study Kähler form by this embedding endows Z with a 2-form φ transversely Kähler with respect to \mathcal{F} [MV]. The form φ defines on X a Kähler form, whose moment polytope is $[-\frac{1}{p+q+\omega_3+\omega_4}, \frac{1}{p+q+\omega_3+\omega_4}]$.

Notice that we can also interpret the choice of ν on the boundary of a chamber: this corresponds to a non simplicial polyhedral decomposition of the vector configuration V (cf. [DL-R-S]).

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